

A CONDITIONAL QUASI-GREEDY BASIS OF l_1

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ABSTRACT. We show that the Lindenstrauss basic sequence in l_1 may be used to construct a conditional quasi-greedy basis of l_1 , thus answering a question of Wojtaszczyk. We further show that the sequence of coefficient functionals for this basis is not quasi-greedy.

1. INTRODUCTION

In what follows $\{e_i\}_{i=1}^\infty$ denotes the standard unit vector basis of l_1 . In [5], the following concept was studied:

Definition. Let \mathfrak{X} be a Banach space with dual \mathfrak{X}^* and let $(x_i, x_i^*)_{i \in F}$ be a fundamental biorthogonal system in $\mathfrak{X} \times \mathfrak{X}^*$ with $\inf_{i \in F} \|x_i\| > 0$ and $\sup_{i \in F} \|x_i^*\| < \infty$. For $m \in \mathbb{N}$, define the operator \mathcal{G}_m by

$$\mathcal{G}_m(x) = \sum_{i \in A} x_i^*(x) x_i,$$

where $A \subset F$ is a set of cardinality m such that $|x_i^*(x)| \geq |x_k^*(x)|$ whenever $i \in A$ and $k \notin A$ (note that the set A depends on x and may not be unique; nevertheless, \mathcal{G}_m is well-defined). Then $(x_i, x_i^*)_{i \in F}$ is a *quasi-greedy system* provided that the operators \mathcal{G}_m satisfy:

$$\lim_m \mathcal{G}_m(x) = x, \quad \text{for each } x \in \mathfrak{X}.$$

Equivalently, by [5, Theorem 1], $(x_i, x_i^*)_{i \in F}$ is a quasi-greedy system if there exists a constant C so that for every $x \in \mathfrak{X}$ and for every $m \in \mathbb{N}$, we have

$$\|\mathcal{G}_m(x)\| \leq C\|x\|.$$

If $\{x_i\}_{i=1}^\infty$ is, in addition, basic, it will be said to be a *quasi-greedy basis*. A sequence $\{z_i\}_{i=1}^\infty$ is *unconditional for constant coefficients* if

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there exist positive numbers c and C such that

$$c \left\| \sum_{i=1}^m z_i \right\| \leq \left\| \sum_{i=1}^m \epsilon_i z_i \right\| \leq C \left\| \sum_{i=1}^m z_i \right\|$$

for any positive integer m and for any sequence of signs $\{\epsilon_i\}_{i=1}^m$.

Remark. By [5, Proposition 2] a quasi-greedy basis is necessarily unconditional for constant coefficients.

In [5], Wojtaszczyk shows that, for $1 < p < \infty$, the space l_p possesses a conditional quasi-greedy basis. Since, by the Remark, a quasi-greedy basis of l_1 is quite close to being equivalent to $\{e_i\}_{i=1}^\infty$, and since $\{e_i\}_{i=1}^\infty$ is the unique normalized unconditional basis of l_1 up to equivalence [3], it is of interest to find a conditional quasi-greedy basis of l_1 . In particular, the existence of such a basis would show that l_1 does not have a unique (up to equivalence) normalized basis that is unconditional for constant coefficients. In this note, we show that indeed such a basis exists.

Our example is derived from the basic sequence constructed by Lindenstrauss in [2]. This is a monotone, conditional basic sequence in l_1 whose closed linear span is a \mathcal{L}_1 -space possessing no unconditional basis. We denote this sequence by $\{x_i\}_{i=1}^\infty$ and its associated sequence of coefficient functionals by $\{x_i^*\}_{i=1}^\infty$. They are defined as follows: For $i \in \mathbb{N}$,

$$x_i = e_i - \frac{1}{2}(e_{2i+1} + e_{2i+2}).$$

Considering x_i^* as an element of $l_\infty/[x_i]^\perp$, we write $x_i^* = y_i^* + [x_i]^\perp$ where the $y_i^* \in l_\infty$ are as defined by Holub and Retherford (see [1]): For each $i \in \mathbb{N}$, let α_i be the finite sequence of positive integers defined by the following conditions:

- 1) $\alpha_i(1) = i$.
- 2) $\alpha_i(j) = \alpha_i(j-1) - ([\alpha_i(j-1)/2] + 1)$ for admissible $j < i$ (that is, such that $\alpha_i(j) > 0$), where $[k]$ denotes the greatest integer less than or equal to k .

Then y_i^* is defined by

$$y_i^* = \sum_{j=1}^{|\alpha_i|} \left(\frac{1}{2}\right)^{j-1} e_{\alpha_i(j)}$$

Thus, for example,

$$\begin{aligned}
y_1^* &= (1, 0, \dots) \\
y_2^* &= (0, 1, 0, \dots) \\
y_3^* &= (\tfrac{1}{2}, 0, 1, 0, \dots) \\
y_4^* &= (\tfrac{1}{2}, 0, 0, 1, 0, \dots) \\
y_5^* &= (0, \tfrac{1}{2}, 0, 0, 1, 0, \dots) \\
y_6^* &= (0, \tfrac{1}{2}, 0, 0, 0, 1, 0, \dots) \\
y_7^* &= (\tfrac{1}{4}, 0, \tfrac{1}{2}, 0, 0, 0, 1, 0, \dots) \\
y_8^* &= (\tfrac{1}{4}, 0, \tfrac{1}{2}, 0, 0, 0, 0, 1, 0, \dots).
\end{aligned}$$

The properties of $\{x_i\}_{i=1}^\infty$ which we shall use in the sequel are summarized in the following fact (see [2], [3], and, also, [4]).

Fact. *The sequence $\{x_i\}_{i=1}^\infty$ satisfies:*

- 1) $\{x_i\}_{i=1}^\infty$ is a monotone basic sequence ([4, p. 455]).
- 2) $[x_i]$ has no unconditional basis ([4, p. 455]).
- 3) For $n \in \mathbb{N}$, there exists an isomorphism T_n from $[x_i : 1 \leq i \leq n]$ onto l_1^n satisfying $\|T_n\| \|T_n^{-1}\| \leq 2$ ([3, Ex. 8.1]).

2. RESULTS

We now construct the basis heralded in the Introduction. To do this, it suffices by [5, Proposition 3] to construct such a basis in a space isomorphic to l_1 . Towards this end, define, for each $n \in \mathbb{N}$,

$$F_n = [x_i : 1 \leq i \leq n].$$

Let

$$\mathfrak{X} = \left(\sum_{i=1}^\infty \oplus F_i \right)_1.$$

We claim that the natural basis $\{\tilde{x}_i\}_{i=1}^\infty$ of \mathfrak{X} obtained from the x_i 's is the desired sequence. Indeed, it follows from the Fact that $\{\tilde{x}_i\}_{i=1}^\infty$ is a monotone, conditional basis of \mathfrak{X} , and that \mathfrak{X} is isomorphic to l_1 . Moreover, assuming for the moment that $\{x_i\}_{i=1}^\infty$ is quasi-greedy, for $m \in \mathbb{N}$ and $y = \sum y_i \in \mathfrak{X}$, we have

$$\|\mathcal{G}_m(y)\| \leq \sup_{k \in K_m} \sum_{i=1}^\infty \|\mathcal{G}_{k(i)}(y_i)\| \leq C \sum_{i=1}^\infty \|y_i\| = C\|y\|,$$

where C is as in the Definition, $K_m = \{k : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\} : \sum_{i=1}^{\infty} k(i) = m\}$, and $\mathcal{G}_0(x) = 0$ for each $x \in \mathfrak{X}$ (note that the operators \mathcal{G}_j appearing in the above inequality are defined with respect to two different sequences). Thus, the fact that $\{\tilde{x}_i\}_{i=1}^{\infty}$ is quasi-greedy will be a consequence of the following theorem.

Theorem. *The sequence $\{x_i\}_{i=1}^{\infty}$ satisfies*

$$(1) \quad 3 \left\| \sum_{i \in S_1 \cup S_2} \alpha_i x_i \right\| \geq \left\| \sum_{i \in S_1} \alpha_i x_i \right\|,$$

whenever S_1 and S_2 are disjoint finite subsets of \mathbb{N} with

$$(2) \quad \min_{i \in S_1} |\alpha_i| \geq \max_{i \in S_2} |\alpha_i|.$$

Proof. For $A \subseteq \mathbb{N}$ and $x \in l_1$, we denote by $P_A x$ the vector in l_1 whose j^{th} -coordinate is $x(j)$ if $j \in A$ and zero otherwise. Let S_1 , S_2 , and $\{\alpha_i\}_{i \in S_1 \cup S_2}$ be as above. Set

$$x = \sum_{i \in S_1} \alpha_i x_i \quad \text{and} \quad y = \sum_{i \in S_2} \alpha_i x_i,$$

and define the sets:

$$A_0 = \left\{ j \in \mathbb{N} : \sum_{i \in S_1} x_i(j) = 1 \right\},$$

$$B_0 = \left\{ j \in \mathbb{N} : \sum_{i \in S_1} x_i(j) = -1/2 \right\},$$

and

$$C_0 = \left\{ j \in \mathbb{N} : \sum_{i \in S_1} x_i(j) = 1/2 \right\}.$$

First, we concentrate our attention on the set B_0 . We define the sets:

$$W_1 = \{ i \in S_2 : x_i(j) = 1 \text{ for some } j \in B_0 \},$$

$$A_1 = \{ j \in A_0 : x_i(j) = -1/2 \text{ for some } i \in W_1 \},$$

and

$$B_1 = \{ j \notin A_0 : x_i(j) = -1/2 \text{ for some } i \in W_1 \}.$$

Finally, set

$$y_0 = x \text{ and } y_1 = \sum_{i \in W_1} \alpha_i x_i.$$

Note that A_1, B_1, B_0 , and C_0 are mutually disjoint. We also have from the triangle inequality that

$$(3) \quad \|P_{B_0} y_0\| \leq \|P_{B_0}(y_0 + y_1)\| + \|P_{B_0} y_1\|.$$

But, since $\|P_{B_0} y_1\| = \|P_{A_1} y_1\| + \|P_{B_1} y_1\|$, we obtain from (3) that

$$(*) \quad \|P_{B_0}(y_0 + y_1)\| + \|P_{B_1} y_1\| \geq \|P_{B_0} y_0\| - \|P_{A_1} y_1\|.$$

Concentrating our attention now on the set B_1 , we let

$$\begin{aligned} W_2 &= \{ i \in S_2 : x_i(j) = 1 \text{ for some } j \in B_1 \}, \\ A_2 &= \{ j \in A_0 : x_i(j) = -1/2 \text{ for some } i \in W_2 \}, \end{aligned}$$

and

$$B_2 = \{ j \notin A_0 : x_i(j) = -1/2 \text{ for some } i \in W_2 \}.$$

Set

$$y_2 = \sum_{i \in W_2} \alpha_i x_i.$$

Then A_1, A_2, B_1, B_2, B_0 , and C_0 are mutually disjoint and, as above, we have

$$(*) \quad \|P_{B_1}(y_1 + y_2)\| + \|P_{B_2} y_2\| \geq \|P_{B_1} y_1\| - \|P_{A_2} y_2\|.$$

In general, at the l^{th} step of the induction, we set

$$\begin{aligned} W_l &= \{ i \in S_2 : x_i(j) = 1 \text{ for some } j \in B_{l-1} \}, \\ A_l &= \{ j \in A_0 : x_i(j) = -1/2 \text{ for some } i \in W_l \}, \end{aligned}$$

and

$$B_l = \{ j \notin A_0 : x_i(j) = -1/2 \text{ for some } i \in W_l \}.$$

Set

$$y_l = \sum_{i \in W_l} \alpha_i x_i.$$

Then the sets A_j, B_j ($1 \leq j \leq l$), B_0 , and C_0 are mutually disjoint, and we have

$$(*) \quad \|P_{B_{l-1}}(y_{l-1} + y_l)\| + \|P_{B_l}y_l\| \geq \|P_{B_{l-1}}y_{l-1}\| - \|P_{A_l}y_l\|.$$

This process must end at some stage k with $W_{k+1} = \emptyset$. Summing the inequalities $(*)$ so obtained and simplifying, we have:

$$(4) \quad \sum_{i=1}^k \|P_{B_{i-1}}(y_{i-1} + y_i)\| + \|P_{B_k}y_k\| \geq \|P_{B_0}y_0\| - \sum_{i=1}^k \|P_{A_i}y_i\|.$$

But, for each $i = 0, 1, \dots, k$, we have $B_i \cap \text{supp } y_j = \emptyset$ for $j \notin \{i, i+1\}$. Moreover, the mutually disjoint sets A_i ($i = 1, 2, \dots, k$) are contained in A_0 . Recalling that $y_0 = x$, from these two observations the inequality (4) reduces to

$$(5) \quad \sum_{i=0}^k \|P_{B_i}(x + y)\| \geq \|P_{B_0}x\| - \|P_{A_0}y\|.$$

Since the sets $C_0, A_0, B_0, B_1, B_2, \dots, B_k$ are mutually disjoint, we have

$$\|x + y\| \geq \|P_{A_0}(x + y)\| + \sum_{i=0}^k \|P_{B_i}(x + y)\| + \|P_{C_0}(x + y)\|;$$

and so, using (5):

$$(6) \quad \|x + y\| \geq \|P_{A_0}(x + y)\| - \|P_{A_0}y\| + \|P_{B_0}x\| + \|P_{C_0}x\|.$$

Using (2) we have $\|P_{A_0}(x + y)\| \geq \|P_{A_0}y\|$, and thus from (6) we have

$$(7) \quad \|x + y\| \geq \|P_{B_0}x\| + \|P_{C_0}x\|.$$

Also, by (2), we have $2\|P_{A_0}(x + y)\| \geq \|P_{A_0}x\|$; thus,

$$(8) \quad 2\|x + y\| \geq \|P_{A_0}x\|.$$

Since $\|x\| = \|P_{A_0}x\| + \|P_{B_0}x\| + \|P_{C_0}x\|$, we may now obtain (1) by adding the inequalities (7) and (8). \square

Remark. We note that the sequence of coefficient functionals for $\{\tilde{x}_i\}_{i=1}^\infty$ is not quasi-greedy. To see this, it is enough to show that $\{x_i^*\}_{i=1}^\infty$ is not quasi-greedy. Towards this end, note that

$$(9) \quad \left\| \sum_{i=1}^{2^{n+1}-2} (-1)^i x_i^* \right\| \leq \left\| \sum_{i=1}^{2^{n+1}-2} (-1)^i y_i^* \right\| = 1.$$

However, defining

$$\{\alpha_n\}_{n=1}^\infty = \{1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots\}$$

and

$$z_n = \sum_{i=1}^{2^{n+1}-2} \alpha_i x_i = e_1 + e_2 + \sum_{i=1}^{2^{n+1}-2} \frac{1}{2^{n+1}} e_{2 \cdot (2^{n+1}-2) - 1 + i},$$

we obtain

$$(10) \quad \left\| \sum_{i=1}^{2^{n+1}-2} x_i^* \right\| \geq \frac{1}{\|z_n\|} \left| \sum_{i=1}^{2^{n+1}-2} x_i^*(z_n) \right| = \frac{1}{4} \left\langle \sum_{i=1}^{2^{n+1}-2} x_i^*, e_1 + e_2 \right\rangle = \frac{n}{2}.$$

It follows from (9), (10), and the Remark of the Introduction that $\{x_i^*\}_{i=1}^\infty$ is not quasi-greedy.

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